Formal Proof of an Exact Scale Invariance in the Nambu-Jona-Lasinio Model

Keiichi Akama

Department of Physics, Saitama Medical College, Kawakado, Moroyama, Saitama, 350-04, Japan

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Abstract

We show that the renormalization group β functions in the Nambu-Jona-Lasinio model identically vanish in all order due to the compositeness condition. Accordingly the effective coupling constants are entirely fixed and do not run with the renormalization scale.

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Quantum fluctuations in some models give rise to composite fields by supplying them the kinetic terms [1,2]. The quantum composites well describe various physical phenomena and are widely applied to the models of hadrons, the composite gauge bosons and composite Higgs scalar [3], collective modes in the nuclear and condensed matter physics, etc.. They can be formulated as the special case of some renormalization theory with the compositeness condition (CC) [4], which says that $Z = 0$, where Z is the wave-function renormalization constant of the to-be-composite field. For example, with CC, the Yukawa model for elementary fermions and bosons reduces to the Nambu-Jona-Lasinio (NJL) type model [1] with elementary fermions and quantum composite bosons. Here we consider the renormalization group (RG) properties of CC in the NJL model [5]. For definiteness, we consider the RG of the 't Hooft type throughout this paper.

The RG analyses with CC in the NJL model have had great impact in phenomenological models [6]. The theoretical aspects of the interplay between RG and CC have been also of continual interest of the people [7]. Most of them, however, considered the limit of the infinite momentum cutoff, which necessitates some additional uncertain assumptions, such as ladder approximation, non-perturbative fixed point, etc., because perturbatively the NJL model is trivial at the limit. Here we take the momentum cutoff as a large but finite physical parameter, and rely on none of theese assumptions. Then we prove the somewhat surprising but important theorem: *In the NJL model, i.e. in the Yukawa model with CC, the RG beta functions identically vanish due to the CC itself, and consequently the coupling constants are scale-invariant.* In the previous paper, we demonstrated it in the RG flow at the leading order in $1/N$ where N is number of the matter fermion species [8]. The derivation suggests that it holds also in higher order. In this paper, we present a simple argument which formally proves the scale invariance *in all order*. This theorem is important because it appears to contradict with the widely spread use of CC and RG in phenomenology [6]. More importantly, however, such a high symmetry is realized in a rather awkward cutoff theory. It is a new type of scale invariance. Furthermore it is an all-order exact result derived from the simple and general assumptions.

We consider the NJL model in its simplest form given by the Lagrangian

$$\mathcal{L}_{\rm N} = \overline{\Psi} i \not\partial \Psi + f |\overline{\Psi}_{\rm L} \Psi_{\rm R}|^2, \tag{1}$$

where $\Psi = (\Psi_1, \Psi_2, \cdots, \Psi_N)$ is a bare color N-plet fermion, f is a bare coupling constant, the subscripts "L" and "R" indicate chiralities. Since the model is not renormalizable, we need to introduce some regularization scheme with a finite cutoff. We adopt the dimensional regularization where we consider everything in $d (= 4 - 2\epsilon)$ dimensional spacetime with small but non-vanishing ϵ. The parameter ϵ roughly corresponds to $1/\log\Lambda$ with the momentum cutoff Λ. The system is equivalent to that described by the Lagrangian [9]

$$\mathcal{L}'_{\rm N} = \overline{\Psi} i \not\partial \Psi + (\overline{\Psi}_{\rm L} \Phi \Psi_{\rm R} + {\rm h.c.}) - \frac{1}{f} |\Phi|^2, \tag{2}$$

where Φ is an auxiliary field.

Now we compare this with the renormalizable Yukawa model for the elementary fermion ψ_0 and the elementary boson ϕ_0 with the following Lagrangian

$$\mathcal{L}_{\rm Y} = \overline{\psi}_0 i \not\partial \psi_0 + g_0 (\overline{\psi}_{0\rm L} \phi_0 \psi_{0\rm R} + {\rm h.c.}) + |\partial_\mu \phi_0|^2 - m_0^2 |\phi_0|^2 - \lambda_0 |\phi_0|^4 \tag{3}$$

where m_0 is the bare mass of ϕ_0, and g_0 and λ_0 are bare coupling constants. To absorb the divergences of the quantum loop diagrams, we renormalize the fields, the mass, and the coupling constants as

$$\psi_0 = \sqrt{Z_\psi} \psi, \quad \phi_0 = \sqrt{Z_\phi} \phi, \quad Z_\phi m_0^2 = Z_m m^2, \tag{4}$$

$$Z_\psi \sqrt{Z_\phi} g_0 = Z_g g \mu^\epsilon, \qquad Z_\phi^2 \lambda_0 = Z_\lambda \lambda \mu^{2\epsilon}, \tag{5}$$

where ψ, ϕ, m, g, and λ are the renormalized fields, mass, and coupling constants, respectively, Z_ψ, Z_ϕ, Z_m, Z_g and Z_λ are the renormalization constants, and μ is a mass scale parameter to make g and λ dimensionless. Then the Lagrangian $\mathcal{L}_{\rm Y}$ becomes

$$\begin{aligned} \mathcal{L}_{\rm Y} = {} & Z_\psi \overline{\psi} i \not\partial \psi + Z_g g \mu^\epsilon (\overline{\psi}_{\rm L} \phi \psi_{\rm R} + {\rm h.c.}) \\ & + Z_\phi |\partial_\mu \phi|^2 - Z_m m^2 |\phi|^2 - Z_\lambda \lambda \mu^{2\epsilon} |\phi|^4. \end{aligned} \tag{6}$$

As the renormalization condition, we adopt the minimal subtraction scheme, where, as the divergent part to be absorbed into in the renormalization constants, we retain all the negative power terms in the Laurent series in ϵ of the divergent (sub)diagrams. Then the parameter μ is interpreted as the renormalization scale. Since the coupling constants are dimensionless, the renormalization constants depend on μ only through g and λ, but do not explicitly depend on μ.

Now we can see that the Lagrangian (6) of the Yukawa model coincides with the Lagrangian (2) of the NJL model, if

$$Z_\phi = Z_\lambda = 0, \quad Z_\psi \neq 0, \quad Z_g \neq 0, \quad Z_m \neq 0. \tag{7}$$

The condition (7) is the so-called "compositeness condition" (CC) [4] which imposes relations among the coupling constants g and λ, the mass m, and the cutoff parameter ϵ in the Yukawa model so that it reduces to the NJL model. In both of the expansion in the coupling constants and that in $1/N$, the perturbative calculations show that $g \to 0$ and $\lambda \to 0$ as $\epsilon \to 0$ at each order, and the theory becomes trivial free theory. Therefore we fix the cutoff $\Lambda = \mu e^{1/\epsilon}$ at some finite value. We can read off from (2) and (6) that the fields and parameters of the NJL and the Yukawa models should be connected by the relations

$$\Psi = \sqrt{Z_\psi}\psi, \quad \Phi = \frac{Z_g g \mu^\epsilon}{Z_\psi}\phi, \quad f = \ Z_g^2 g^2 \mu^{2\epsilon}/Z_\psi^2 Z_m m^2. \tag{8}$$

The last of (8) is so-called "gap equation" of the NJL model. In terms of the bare parameters, the CC (7) corresponds to the limit

$$g_0 \to \infty, \qquad \lambda_0/g_0^4 \to 0. \tag{9}$$

These behaviors may look singular at first sight, but they are of no harm because they are unobservable bare quantities.

Thus the NJL model is equivalent to the cutoff Yukawa model (i.e. the Yukawa model with a finite cutoff) under the CC (7). Then the RG of the former coincides with that of the latter under the CC (7). Let us consider the latter (the cutoff Yukawa model) with special

cares on the finite cutoff. In our case, it amounts to fix $\epsilon = (4-d)/2$ at some non-vanishing value. The beta functions and the anomalous dimensions are defined as

$$\beta_g^{(\epsilon)}(g,\lambda) = \mu\frac{\partial g}{\partial\mu}\ , \qquad \beta_\lambda^{(\epsilon)}(g,\lambda) = \mu\frac{\partial\lambda}{\partial\mu}\ , \tag{10}$$

$$\gamma_\phi^{(\epsilon)}(g,\lambda) = \frac{1}{2}\mu\frac{\partial\ln Z_\phi}{\partial\mu}\ , \qquad \gamma_\psi^{(\epsilon)}(g,\lambda) = \frac{1}{2}\mu\frac{\partial\ln Z_\psi}{\partial\mu}\ , \tag{11}$$

where the differentiation $\partial/\partial\mu$ performed with g_0, λ_0, and ϵ fixed. Operating $\mu(\partial/\partial\mu)$ to the equations in (5) we obtain

$$\left[\beta_g^{(\epsilon)}\frac{\partial}{\partial g} + \beta_\lambda^{(\epsilon)}\frac{\partial}{\partial\lambda} + \epsilon\right] gJ = 0, \qquad \left[\beta_g^{(\epsilon)}\frac{\partial}{\partial g} + \beta_\lambda^{(\epsilon)}\frac{\partial}{\partial\lambda} + 2\epsilon\right]\lambda K = 0, \tag{12}$$

where $J = Z_g/(Z_\psi\sqrt{Z_\phi})$ and $K = Z_\lambda/Z_\phi^2$. Comparing the residues of the poles at $\epsilon = 0$, we obtain

$$\beta_g^{(\epsilon)} = -\epsilon g + g\mathcal{D}J_1, \qquad \beta_\lambda^{(\epsilon)} = -2\epsilon\lambda + \lambda\mathcal{D}K_1, \tag{13}$$

where $\mathcal{D} = g(\partial/\partial g) + 2\lambda(\partial/\partial\lambda)$, and J_1 and K_1 are the residues of the simple poles of J and K, respectively. On the other hand the anomalous dimensions are given by

$$\gamma_\phi^{(\epsilon)} = -\frac{1}{2}\mathcal{D}Z_{\phi 1}, \qquad \gamma_\psi^{(\epsilon)} = -\frac{1}{2}\mathcal{D}Z_{\psi 1}, \tag{14}$$

where $Z_{\phi 1}$ and $Z_{\psi 1}$ are the residues of the simple poles of Z_ϕ and Z_ψ, respectively. We can read off from (13) and (14) that $\beta^{(\epsilon)}$'s depend on the cutoff only through the first terms $-\epsilon g$ and $-2\epsilon\lambda$ of the expressions, while $\gamma^{(\epsilon)}$'s are independent of ϵ. We should be careful not to neglect the cutoff dependence of $\beta^{(\epsilon)}$'s.

For an illustration, we start with the leading-order approximation in $1/N$. Explicit calculations give

$$Z_\phi = 1 - \frac{Ng^2}{16\pi^2\epsilon}\ , \quad Z_\lambda = 1 - \frac{Ng^4}{16\pi^2\epsilon\lambda}\ , \quad Z_g = Z_\psi = 1. \tag{15}$$

Using (13) and (14), we get

$$\beta_g^{(\epsilon)} = -\epsilon g + \frac{Ng^3}{16\pi^2}\ , \quad \beta_\lambda^{(\epsilon)} = -2\epsilon\lambda + \frac{N(4g^2\lambda - 2g^4)}{16\pi^2}\ , \tag{16}$$

$$\gamma_\phi^{(\epsilon)} = \frac{Ng^2}{16\pi^2}\ , \quad \gamma_\psi^{(\epsilon)} = 0. \tag{17}$$

The CC $Z_\phi = Z_\lambda = 0$ in (7) with (15) is solved to give

$$g^2 = \lambda = \frac{16\pi^2\epsilon}{N} \ . \tag{18}$$

Then we substitute (18) into (16) to get $\beta_g^{(\epsilon)} = \beta_\lambda^{(\epsilon)} = 0$. Thus the statement that the beta functions in the NJL model vanish is proved at this order. The coupling constants g and λ do not run with the scale μ. In the previous paper, we demonstrated the scale invariance of the NJL model in the renormalization group flow of the general case [8].

If we consider coupling-constant expansion or loop expansion instead of the $1/N$ expansion, the $\beta^{(\epsilon)}$'s appear to fail to vanish. It is, however, because the former expansions (the coupling-constant and the loop expansions) are inconsistent in the NJL model as follows. At the leading order in these expansions, the CC takes the same form as (18). The leading-order contributions to $\beta^{(\epsilon)}$ involve the diagrams with boson lines. A one-fermion-loop insertion into a boson line give rise to a extra factor of $O(g^2N/\epsilon)$, which is order unity according to (18). Then the infinitely many higher-order diagrams with one-fermion-loop insertions have the same order of magnitude. Therefore the coupling-constant and the loop expansions fail, The suitable expansion is that in $1/N$.

Thus the NJL model is at a fixed point in the RG flow of the Yukawa model at this order. The coupling constants in the NJL model are scale-invariant, and do not run with the scale parameter. We can trace back the reason of scale invariance to the fact that beta functions vanish due to the compositeness condition. It is further traced back to the fact that the scale invariance of the relation (5) under the compositeness condition (7). Thus we expect that the scale invariance holds not only in the leading order in $1/N$, but also in all order.

Now we show that, under the compositeness condition, the beta functions vanish in all order. The dependence of g and λ on ε and μ should be determined by the RG equations (10) with the beta functions in (13). They are derived from eq. (5), and it is eq. (5) that originally determines the ε- and μ-dependence of g and λ. We rewrite (5) into the form

$$F \equiv Z_\phi - \frac{Z_g^2 g^2}{Z_\psi^2}\,\frac{\mu^{2\varepsilon}}{g_0^2} = 0 \ , \quad G \equiv Z_\lambda - \frac{Z_g^4 g^4}{Z_\psi^4 \lambda}\,\frac{\lambda_0 \mu^{2\varepsilon}}{g_0^4} = 0 \ . \tag{19}$$

Note that Z's are functions of g, λ, and ε, and consequently F and G are functions of g, λ, ε, μ, g_0, and λ_0. A mathematical theorem says that, if $\partial(F, G)/\partial(g, \lambda) \neq 0$ in an appropriate region, the equation (19) has a unique solution for g and λ as functions of ε, μ, g_0, and λ_0, and

$$\begin{pmatrix} \frac{\partial g}{\partial \mu} \\ \frac{\partial \lambda}{\partial \mu} \end{pmatrix} = - \begin{pmatrix} \frac{\partial F}{\partial g} & \frac{\partial F}{\partial \lambda} \\ \frac{\partial G}{\partial g} & \frac{\partial G}{\partial \lambda} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F}{\partial \mu} \\ \frac{\partial G}{\partial \mu} \end{pmatrix}. \tag{20}$$

It is sufficient for us to consider the case where the CC (7) has a nontrivial solution. Therefore $\partial(Z_\phi, Z_\lambda)/\partial(g, \lambda) \neq 0$, $g \neq 0$, and $\lambda \neq 0$. Then, by (19), the first matrix in r.h.s. of (20) is finite in the NJL limit (9). On the other hand, in the same limit, we have

$$\frac{\partial F}{\partial \mu} = -\frac{Z_g^2 g^2}{Z_\psi^2} \frac{2\varepsilon\mu^{2\varepsilon-1}}{g_0^2} \to 0, \qquad \frac{\partial G}{\partial \mu} = -\frac{Z_g^4 g^4}{Z_\psi^4 \lambda} \frac{2\varepsilon\lambda_0\mu^{2\varepsilon-1}}{g_0^4} \to 0, \tag{21}$$

Therefore, from (20), $\partial g/\partial\mu \to 0$ and $\partial\lambda/\partial\mu \to 0$, and hence $\beta_g^{(\varepsilon)} \to 0$ and $\beta_\lambda^{(\varepsilon)} \to 0$ in the compositeness limit. Thus, *the compositeness condition implies scale invariance of the coupling constants in all order*.

What is newly proved here is that the NJL model is always just on the fixed point of the cutoff Yukawa model in the all order. The often claimed statement "there is 'a' nontrivial infrared stable fixed point in the NJL model" [5] is inappropriate because no other possibility is allowed than the fixed point. The scale invariant (Yukawa) coupling constant is not "a" solution, but the only solution in the NJL model. It is fixed even in the ultraviolet region. We can demonstrate it in lower orders in $1/N$ by using the compositeness condition which characterizes the NJL model in the cutoff Yukawa model [8]. Here it is proved in the all order.

So far we have illustrated the theorem in terms of the simplest NJL model. It is, however, obvious that the theorem can be extended to more general cases including e.g. a vectorial composite [10], a boson-boson or a boson-fermion composite. It would be also applicable to the composite gravity [11] and the brane induced gravity [12], by using renormalizable R^2 gravity, if we ignore some drawbacks. It is surprising that the non-renormalizable NJL model with a finite cutoff shares the property of vanishing β function with highly symmetric

models like e.g. the $N = 4$ super-Yang-Mills theory. The CC of the NJL-type model is so strong as to impose such a high symmetry.

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